

LIE ISOMORPHISMS OF DERIVED RINGS OF SIMPLE RINGS⁽¹⁾

BY

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1. Introduction. A Lie subring L of an associative ring R is an additive subgroup of R such that $[x, y] = xy - yx \in L$, whenever x and y are in L . Clearly $[R, R]$, the additive subgroup of R generated by all commutators $[x, y]$, is such a Lie subring of R . If L_1 is a Lie subring of R and L_2 is a Lie subring of S , then a Lie isomorphism ϕ of L_1 onto L_2 is a one-one additive mapping of L_1 onto L_2 which preserves commutators, i.e.

$$\begin{aligned}\phi(x+y) &= \phi(x) + \phi(y) \\ \phi(xy-yx) &= \phi(x)\phi(y) - \phi(y)\phi(x)\end{aligned}$$

for all $x, y \in L_1$. In this paper, we will assume that $L_1 = [R, R]$ and $L_2 = [S, S]$ where R and S are simple rings with identity. We shall also assume that the characteristic of R is different from 2 and 3, and that R contains three nonzero orthogonal idempotents whose sum is the identity. We will then show that ϕ may be extended to either an isomorphism of R onto S , or to the negative of an anti-isomorphism of R onto S . This result generalizes a theorem of Martindale [4, p. 916, Theorem 5].

2. Lie isomorphisms and the Peirce decomposition. Let e_1, e_2 , and e_3 be the orthogonal idempotents of R , i.e.

$$e_i^2 = e_i \neq 0; \quad \sum_{i=1}^3 e_i = 1, \quad e_i e_j = 0 \text{ for } i \neq j.$$

It is well known that we can obtain the Peirce decomposition

$$R = \bigoplus_{i,j=1}^3 R_{ij} \quad \text{where } R_{ij} = e_i R e_j.$$

We will denote an element in R_{ij} by x_{ij} . The proof of the theorem requires a careful analysis of those properties of the Peirce decomposition, which are invariant under Lie isomorphisms.

Let S be a simple ring with identity. Let S_r and S_l denote the right and left multiplications respectively of S , and denote the center of S by Z .

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2.1. LEMMA.

$$S^* \otimes_Z S \cong S_l S_r.$$

Proof. Let $\eta: S^* \otimes_Z S \rightarrow S_l S_r$ be given by

$$\left(\sum_{i=1}^n a_i^* \otimes b_i \right) \eta = \sum_{i=1}^n a_{il} b_{ir}.$$

Since $(1^* \otimes 1)\eta = 1$, we know $\eta \neq 0$. Since $S^* \otimes_Z S$ is simple, η is an isomorphism, and η is clearly a surjection.

The following lemma illustrates how one can solve certain “generalized polynomial identities” using the tensor product.

2.2. LEMMA. *Let S be a simple ring with identity of characteristic not 2 or 3, such that $[S, S]^- = S$, where $[S, S]$ denotes the subring generated by $[S, S]$. Suppose $[[[x, a], a], a] = 0$ for all $x \in [S, S]$. Then there is $z \in Z$ such that $(a+z)^2 = 0$.*

Proof. Since $[[[x, a], a], a] = 0$ for all $x \in [S, S]$, we may choose $x = [y, a]$ where y is arbitrary in S . Hence $[[[[y, a], a], a], a] = 0$ for all $y \in S$. In terms of mappings, this gives that $(a_r - a_l)^4 = 0$. Since $[a_r, a_l] = 0$, we can expand the previous relation to obtain

$$a_r^4 - 4a_r^3 a_l + 6a_r^2 a_l^2 - 4a_r a_l^3 + a_l^4 = 0.$$

By 2.1 we may replace this equation by:

$$(1) \quad 1 \otimes a^4 - 4a \otimes a^3 + 6a^2 \otimes a^2 - 4a^3 \otimes a + a^4 \otimes 1 = 0.$$

Since $1 \neq 0$, the set $\{a^4, a^3, a^2, a, 1\}$ is a dependent set over Z . We may assume that $\{a^3, a^2, a, 1\}$ is a dependent set. Otherwise

$$a^4 = \alpha a^3 + \beta a^2 + \gamma a + \delta, \quad \alpha, \beta, \gamma, \delta \in Z.$$

Substituting this in (1), we obtain:

$$(\alpha - 4a) \otimes a^3 + (\beta + 6a^2) \otimes a^2 + (\gamma - 4a^3) \otimes a + (\alpha a^3 + \beta a^2 + \gamma a + \delta) \otimes 1 = 0.$$

The independence of $\{a^3, a^2, a, 1\}$ gives that $\alpha - 4a = 0$. But then $a \in Z$ and $z = -a$ satisfies the theorem. We now claim that the set $\{a^2, a, 1\}$ is a dependent set. If this is not the case, then we have:

$$a^3 = \alpha a^2 + \beta a + \gamma \quad \alpha, \beta, \gamma \in Z$$

whence

$$a^4 = (\alpha^2 + \beta)a^2 + (\alpha\beta + \gamma)a + \alpha\gamma.$$

These relations, when substituted into (1), give

$$\begin{aligned} & (-6a^2 - 4\alpha a + \alpha^2 + \beta) \otimes a^2 + (\alpha a^2 - 3\beta a + 2\gamma + \alpha\beta) \otimes a \\ & + [(a^2 + \beta)a^2 + (\alpha\beta - 3\gamma)a + 2\alpha\gamma] \otimes 1 = 0. \end{aligned}$$

The assumed independence of $\{a^2, a, 1\}$ gives that $-6a^2 - 4\alpha a + \alpha^2 + \beta = 0$ which contradicts the independence of $\{a^2, a, 1\}$. Thus $\{a^2, a, 1\}$ is a dependent set as

claimed. Furthermore, if $\{a, 1\}$ is dependent, then $a \in Z$ and $z = -a$ satisfies the theorem. If $\{a, 1\}$ is independent, then we have that:

$$(2) \quad a^2 = \alpha a + \beta$$

whence

$$(3) \quad a^3 = (\alpha^2 + \beta)a + \beta\alpha.$$

But $[[[x, a], a], a] = 0$ for all $x \in [S, S]$, so

$$xa^3 - 3axa^2 + 3a^2xa - a^3x = 0 \quad \text{for all } x \in [S, S].$$

Substituting the relations (2) and (3) this equation becomes after simplification

$$(4) \quad (\alpha^2 + 4\beta)[x, a] = 0 \quad \text{for all } x \in [S, S].$$

Now, if $[x, a] = 0$ for all $x \in [S, S]$, then, since $[S, S]^- = S$, $a \in Z$ and we are done as before. If $[x, a] \neq 0$ for some $x \in [S, S]$, then, since Z is a field, $\alpha^2 + 4\beta = 0$. Let $z = -\alpha/2$. Now $(a - \alpha/2)^2 = a^2 - \alpha a + \alpha^2/4 = a^2 - \alpha a - \beta = 0$.

2.3. LEMMA. *Let S be simple with identity and with characteristic different from 2. Suppose $a, b \in S$ are such that $a^2 = b^2 = [a, b] = 0$. If, in addition, $[[[x, b], a], b] = 0$ for all $x \in [S, S]$, then $ab = ba = 0$.*

Proof. Since $[[[x, b], a], b] = 0$ for all $x \in [S, S]$, letting $x = [y, a]$ where y is arbitrary in S , we have $[[[[y, a], b], a], b] = 0$ for all $y \in S$. Expanding this equation and using $a^2 = b^2 = [a, b] = 0$, we obtain $4abyab = 0$ for all $y \in S$. Since S is simple, $ab = 0$.

2.4. LEMMA. *Let S be a simple ring such that $[S, S]^- = S$. Suppose further that $a[S, S]b = 0$ for some $a, b \in S$. Then either $a = 0$ or $b = 0$.*

Proof. Let $x, y \in S$. Since $xy - yx \in [S, S]$, we have $a(xy - yx)b = 0$ or $axyb = ayxb$. Now let $L = \{x \in S \mid xb = 0\}$. L is a left ideal of S and $a[S, S] \subseteq L$. LS is a two-sided ideal of S , and so either $LS = 0$ or $LS = S$. If $LS = 0$, then $L = 0$, so $a[S, S] = 0$. Since $[S, S]^- = S$, this gives $aS = 0$ and hence $a = 0$. Hence we may assume that $LS = S$. Let $x \in S$, then $x = \sum_{i=1}^n l_i y_i$ where $l_i \in L$ and $y_i \in S$. Then

$$axb = a\left(\sum_{i=1}^n l_i y_i\right)b = \sum_{i=1}^n a(l_i y_i)b = \sum_{i=1}^n (ay_i l_i b) = 0.$$

Thus $aSb = 0$, so either $a = 0$ or $b = 0$.

We now state in the form of a remark a useful result which may be found in [1].

REMARK 2.5. If R is a simple ring of characteristic different from 2 and is not a field, then $[R, R]^- = R$.

Henceforth R and S will be as stated in the introduction. The "off-diagonal" elements R_{ij} , $i \neq j$ of the Peirce decomposition of R are in $[R, R]$. In fact $x_{ij} = [e_i, x_{ij}]$.

REMARK 2.6. The characteristic of S is not two or three, and $([S, S])^- = S$.

Proof. Since the ideal $\{x \in R \mid 2x=0\}$ must be zero, $2R_{12} \neq 0$. Thus $2\phi(R_{12}) \neq 0$ and $2S \neq 0$. Similarly the characteristic of S is not three.

By 2.5 $[R, R]^- = R$, so $[R, R] \neq 0$. Since ϕ is a bijection $[S, S] \neq 0$. Thus by 2.5 $[S, S]^- = S$.

We now begin to examine the image of the Peirce decomposition under ϕ . If $x_{ij} \in R_{ij}$, $i \neq j$, then $x_{ij} \in [R, R]$. Thus ϕ may be applied to these elements.

2.7. LEMMA. Let $x_{ij} \in R_{ij}$, $i \neq j$. Then $\phi(x_{ij})^2 = 0$.

Proof. If $x_{ij} = 0$, then $\phi(x_{ij})^2 = 0$. So we may assume that $x_{ij} \neq 0$. Since $x_{ij}^2 = 0$, $[[[x, x_{ij}], x_{ij}], x_{ij}] = 0$ for all $x \in [R, R]$. Because ϕ is a Lie isomorphism this gives $[[[\phi(x), \phi(x_{ij})], \phi(x_{ij})], \phi(x_{ij})] = 0$. But ϕ is a surjection, so

$$[[[x, \phi(x_{ij})], \phi(x_{ij})], \phi(x_{ij})] = 0$$

for all $x \in [S, S]$. By 2.2, $\phi(x_{ij}) = b + \lambda$, where $\lambda \in Z(S)$ and $b^2 = 0$. This is true for all i, j where $i \neq j$. Furthermore, $b \neq 0$. Otherwise $\phi(x_{ij}) \in Z$, so $[\phi(x_{ij}), \phi(x_{jk})] = 0$ for $k \neq i$, $k \neq j$. Since ϕ is a Lie isomorphism, this gives $\phi([x_{ij}, x_{jk}]) = 0$. Hence $[x_{ij}, x_{jk}] = 0$, so $x_{ij}x_{jk} = 0$. But then $x_{ij}R_{jk} = 0$. Hence $x_{ij} = 0$, a contradiction.

For convenience in notation, let us assume that $i=1$ and $j=2$, that is, we wish to show that $\phi(x_{12})^2 = 0$. For this purpose let $y_{13} \in R_{13}$, $y_{12} \in R_{12}$, and $y_{32} \in R_{32}$ be arbitrary nonzero elements such that $y_{13}y_{32} \neq 0$. By the above argument we have:

- (a) $\phi(y_{13}) = b + \lambda$, $b^2 = 0$, $b \neq 0$, $\lambda \in Z(S)$,
- (b) $\phi(y_{12}) = c + \mu$, $c^2 = 0$, $c \neq 0$, $\mu \in Z(S)$,
- (c) $\phi(y_{13}y_{32}) = d + \nu$, $d^2 = 0$, $d \neq 0$, $\nu \in Z(S)$.

Now $[[[x, y_{13}], y_{12}], y_{13}] = 0$ for all $x \in [R, R]$, thus

$$[[[x, \phi(y_{13})], \phi(y_{12})], \phi(y_{13})] = 0 \quad \text{for all } x \in [S, S].$$

Since $\lambda, \mu \in Z(S)$, this gives $[[[x, b], c], b] = 0$. Since $[y_{13}, y_{12}] = 0$, $[\phi(y_{13}), \phi(y_{12})] = 0$, and so $[b, c] = 0$. By 2.3 $bc = cb = 0$. Hence,

$$(1) \quad \phi(y_{13})c = (b + \lambda)c = \lambda c.$$

Since $[y_{32}, y_{12}] = 0$, $[\phi(y_{32}), \phi(y_{12})] = 0$ and so $[\phi(y_{32}), c] = 0$. Commuting (1) with $\phi(y_{32})$, we obtain

$$(2) \quad [\phi(y_{13})c, \phi(y_{32})] = [\lambda c, \phi(y_{32})] = 0.$$

Since $[\phi(y_{13})c, \phi(y_{32})] = [\phi(y_{13}), \phi(y_{32})]c = \phi([y_{13}, y_{32}])c$, we have

$$(3) \quad \phi(y_{13}y_{32})c = 0.$$

But then from (c),

$$(4) \quad (d + \nu)c = 0.$$

An application of 2.3 shows that $dc = 0$, hence

$$(5) \quad \nu c = 0.$$

$$(6) \quad \nu = 0.$$

We have shown that $\phi(y_{13}y_{32})^2=0$. Since $R_{12}=R_{13}R_{32}$, we may write

$$x_{12} = \sum_{i=1}^n y_{13}^{(i)} y_{32}^{(i)}.$$

Hence $\phi(x_{12}) = \sum_{i=1}^n \phi(y_{13}^{(i)} y_{32}^{(i)})$. We have just shown that $\phi(y_{13}^{(i)} y_{32}^{(i)})^2=0$. Because

$$[\phi(y_{13}^{(i)} y_{32}^{(i)}), \phi(y_{13}^{(j)} y_{32}^{(j)})] = 0$$

and

$$[[[x, \phi(y_{13}^{(i)} y_{32}^{(i)})], \phi(y_{13}^{(j)} y_{32}^{(j)})], \phi(y_{13}^{(i)} y_{32}^{(i)})] = 0,$$

we have by 2.3 that $\phi(y_{13}^{(i)} y_{32}^{(i)}) \phi(y_{13}^{(j)} y_{32}^{(j)}) = 0$. Thus

$$\phi(x_{12})^2 = \left(\sum_{i=1}^n \phi(y_{13}^{(i)} y_{32}^{(i)}) \right)^2 = 0.$$

2.8. LEMMA. *Let $x_{ij} \in R_{ij}$, $x_{kl} \in R_{kl}$ where $i \neq j$ and $k \neq l$. If $x_{kl}x_{ij} = x_{ij}x_{kl} = 0$, then $\phi(x_{ij})\phi(x_{kl}) = \phi(x_{kl})\phi(x_{ij}) = 0$.*

Proof. Since $[[[x, x_{ij}], x_{kl}], x_{ij}] = 0$ for all $x \in [R, R]$ and $[x_{ij}, x_{kl}] = 0$, we have

$$[[[x, \phi(x_{ij})], \phi(x_{kl})], \phi(x_{ij})] = 0 \quad \text{for all } x \in [S, S],$$

and $[\phi(x_{ij}), \phi(x_{kl})] = 0$. Furthermore $\phi(x_{ij})^2 = 0$ and $\phi(x_{kl})^2 = 0$. Hence by 2.3 $\phi(x_{ij})\phi(x_{kl}) = 0$.

In order to continue the study of the Peirce decomposition under a Lie isomorphism, we must examine the relationship between $[R, R]$ and the "off-diagonal" components R_{ij} , $i \neq j$. To this end we have:

2.9. LEMMA. *$[R, R]$ is additively generated by R_{ij} , $i \neq j$, and $[R_{ij}, R_{ji}]$ for $i \neq j$.*

Proof.

$$\begin{aligned} [R, R] &= \left[\bigoplus_{i,j=1}^3 R_{ij}, \bigoplus_{i,j=1}^3 R_{ij} \right] \\ &= \sum_{i \neq j} R_{ij} + \sum_{i \neq j} [R_{ij}, R_{ji}] + \sum_{i=1}^3 [R_{ii}, R_{ii}]. \end{aligned}$$

Thus we need only show that $[R_{ii}, R_{ii}] \subseteq [R_{ij}, R_{ji}]$ for $i \neq j$. Without loss of generality, we may assume that $i=1$ and $j=2$. Then $R_{11} = e_1 R e_1 = e_1 R e_2 R e_1$. Let $x, y \in R_{11}$. Write $x = \sum_{i,j} e_1 x_i e_2 y_j e_1$ and $y = e_1 w e_1$. Then

$$[x, y] = \left[\sum_{i,j} e_1 x_i e_2 y_j e_1, e_1 w e_1 \right] = \sum_{i,j} [e_1 x_i e_2 y_j e_1, e_1 w e_1].$$

So it suffices to show that $[e_1 x_i e_2 y_j e_1, e_1 w e_1] \in [R_{12}, R_{21}]$. But

$$[e_1 x_i e_2 y_j e_1, e_1 w e_1] = [e_1 x_i e_2, e_2 y_j e_1 w e_1] - [e_1 w e_1 x_i e_2, e_2 y_j e_1]$$

which is in $[R_{12}, R_{21}]$.

Since ϕ is a Lie isomorphism, 2.9 can be carried over to S .

2.10. LEMMA. $[S, S]$ is additively spanned by $\phi(x_{ij})$, $i \neq j$, and $\phi[x_{ij}, x_{ji}]$ for $i \neq j$.

Proof. The result is immediate from 2.9 and the fact that ϕ is a surjection.

We are trying to show that ϕ can be extended to either an isomorphism or the negative of an anti-isomorphism. Lemma 2.7 hints that ϕ is well-behaved. The next lemma, which is the key to the main theorem, determines ϕ on certain of the off-diagonal components.

2.11. LEMMA. Let (i, j, k) be any permutation of $(1, 2, 3)$. Suppose $x_{ij} \in R_{ij}$ and $x_{jk} \in R_{jk}$. Then either:

- (1) $\phi(x_{ij}x_{jk}) = \phi(x_{ij})\phi(x_{jk})$, or
- (2) $\phi(x_{ij}x_{jk}) = -\phi(x_{jk})\phi(x_{ij})$.

Proof. Without loss of generality we may assume $i=1$, $j=2$, and $k=3$. The method of proof will be to show that

$$\phi(x_{12})\phi(x_{23})[S, S]\phi(x_{23})\phi(x_{12}) = 0.$$

That this suffices, can be seen as follows:

By 2.4 either $\phi(x_{12})\phi(x_{23})=0$ or $\phi(x_{23})\phi(x_{12})=0$. Since ϕ is a Lie isomorphism,

$$\phi(x_{12}x_{23}) = \phi([x_{12}, x_{23}]) = [\phi(x_{12}), \phi(x_{23})] = \phi(x_{12})\phi(x_{23}) - \phi(x_{23})\phi(x_{12}).$$

This gives the result.

Since $[S, S]$ is additively spanned by elements of the form $\phi(x_{ij})$, $i \neq j$ and $[\phi(x_{ij}), \phi(x_{ji})]$, $i \neq j$, it suffices to consider these elements only.

$$\begin{aligned} \phi(x_{12})\phi(x_{23})\phi(y_{12})\phi(x_{23})\phi(x_{12}) &= \phi(x_{12})\phi(x_{23})\phi(y_{12})\phi(x_{23})\phi(x_{12}) \\ &\quad - \phi(x_{12})\phi(y_{12})\phi(x_{23})\phi(x_{23})\phi(x_{12}) \quad (\text{by 2.7}) \\ &= \phi(x_{12})\phi(x_{23}y_{12} - y_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\ &\quad (\text{since } \phi \text{ is a Lie isomorphism}) \\ &= -\phi(x_{12})\phi(y_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\ &= 0 \quad (\text{since by 2.8, } \phi(x_{12})\phi(y_{12}x_{23}) = 0). \end{aligned}$$

$$(2) \quad \phi(x_{12})\phi(x_{23})\phi(y_{ij})\phi(x_{23})\phi(x_{12}) = 0, \quad \text{for } (i, j) = (1, 3), (2, 1), (2, 3) \quad (\text{by 2.8}).$$

$$\begin{aligned} \phi(x_{12})\phi(x_{23})\phi(y_{31})\phi(x_{23})\phi(x_{12}) &= \phi(x_{12})\phi(x_{23})\phi(y_{31})\phi(x_{23})\phi(x_{12}) \\ &\quad - \phi(x_{12})\phi(y_{31})\phi(x_{23})\phi(x_{23})\phi(x_{12}) \\ &= \phi(x_{12})\phi(x_{23}y_{31} - y_{31}x_{23})\phi(x_{23})\phi(x_{12}) \\ &\quad (\text{since } \phi \text{ is a Lie isomorphism}) \\ &= \phi(x_{12})\phi(x_{23}y_{31})\phi(x_{23})\phi(x_{12}) \\ &= 0 \quad (\text{since by 2.8, } \phi(x_{23}y_{31})\phi(x_{23}) = 0). \end{aligned}$$

$$\begin{aligned}
(4) \quad & \phi(x_{12})\phi(x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) = \phi(x_{12})\phi(x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
& \quad - \phi(x_{23})\phi(x_{12})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
& = \phi(x_{12}x_{23} - x_{23}x_{12})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
& \quad \text{(since } \phi \text{ is a Lie isomorphism)} \\
& = \phi(x_{12}x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
& = \phi(x_{12}x_{23})\phi(y_{32})\phi(x_{23})\phi(x_{12}) \\
& \quad - \phi(y_{32})\phi(x_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\
& = \phi(x_{12}x_{23}y_{32} - y_{32}x_{12}x_{23})\phi(x_{23})\phi(x_{12}) \\
& = \phi(x_{12}x_{23}y_{32})\phi(x_{23})\phi(x_{12}) \\
& = \phi(x_{12}x_{23}y_{32})\phi(x_{23})\phi(x_{12}) \\
& \quad - \phi(x_{23})\phi(x_{12}x_{23}y_{32})\phi(x_{12}) \\
& = \phi(x_{12}x_{23}y_{32}x_{23})\phi(x_{12}) \\
& = 0 \quad (\text{by 2.8}).
\end{aligned}$$

$$(5) \quad \phi(x_{12})\phi(x_{23})([\phi(y_{ij}), \phi(y_{ji})])\phi(x_{23})\phi(x_{12}) = 0 \quad \text{for } i \neq j \quad \text{by 2.8.}$$

This concludes the proof.

This lemma points the way of the main theorem. It says that ϕ is either an associative isomorphism or the negative of an anti-isomorphism on certain parts of R . It is most convenient to break the proof of the main theorem into two cases depending on the outcome of this lemma. The two cases will occupy §§3 and 4 respectively.

3. The isomorphism case. In this section we will continue the proof of the theorem under the following assumption:

3.1. ASSUMPTION. There is $r_{12} \in R_{12}$, $r_{23} \in R_{23}$ such that $r_{12}r_{23} \neq 0$ and

$$\phi(r_{12}r_{23}) = \phi(r_{12})\phi(r_{23}).$$

The first task is to show that 3.1 determines the behavior of ϕ on all products $y_{12}y_{23}$.

3.2. LEMMA. If $y_{12} \in R_{12}$, $y_{23} \in R_{23}$, then $\phi(y_{12}y_{23}) = \phi(y_{12})\phi(y_{23})$.

Proof. We may assume $y_{12}y_{23} \neq 0$. Otherwise, $\phi(y_{12}y_{23}) = 0$. But then

$$0 = \phi[y_{12}, y_{23}] = \phi(y_{12})\phi(y_{23}) - \phi(y_{23})\phi(y_{12}).$$

Thus $\phi(y_{12})\phi(y_{23}) = \phi(y_{23})\phi(y_{12})$. But by 2.11 one of these terms is zero. Hence $0 = \phi(y_{12}y_{23}) = \phi(y_{12})\phi(y_{23})$.

Now suppose the lemma is false. Then $\phi(y_{12}y_{23}) = -\phi(y_{23})\phi(y_{12})$ by 2.11.

We claim $r_{12}y_{23} = 0$. For this, consider

$$(1) \quad \phi(r_{12}(y_{23} + r_{23})) = \phi(r_{12}y_{23}) + \phi(r_{12}r_{23}).$$

By 2.11, $\phi(r_{12}y_{23}) = \phi(r_{12})\phi(y_{23})$ or $\phi(r_{12}y_{23}) = -\phi(y_{23})\phi(r_{12})$. Suppose

$$(2) \quad \phi(r_{12}y_{23}) = -\phi(y_{23})\phi(r_{12}).$$

Then from (1) we have

$$(3) \quad \phi(r_{12}(y_{23} + r_{23})) = -\phi(y_{23})\phi(r_{12}) + \phi(r_{12})\phi(r_{23}).$$

On the other hand, by 2.11 either

$$(4) \quad \phi(r_{12}(y_{23} + r_{23})) = \phi(r_{12})\phi(y_{23} + r_{23})$$

or

$$(5) \quad \phi(r_{12}(y_{23} + r_{23})) = -\phi(y_{23} + r_{23})\phi(r_{12}).$$

If (4) is true, then using (3) we obtain

$$(6) \quad -\phi(y_{23})\phi(r_{12}) = \phi(r_{12})\phi(y_{23}).$$

It is immediate from (6) that $r_{12}y_{23} = 0$, and the claim is true. If (5) is true, then again from (3) we obtain

$$(7) \quad \phi(r_{12})\phi(r_{23}) = -\phi(r_{23})\phi(r_{12}).$$

It follows from (7) that $r_{12}r_{23} = 0$, a contradiction. Thus if (2) is true, the claim has been proven. If (2) is false, then we have

$$(8) \quad \phi(r_{12}y_{23}) = \phi(r_{12})\phi(y_{23}).$$

Reasoning as before, we find that either $y_{12}y_{23} = 0$ or $r_{12}y_{23} = 0$. Hence $r_{12}y_{23} = 0$.

We claim also that $y_{12}r_{23} = 0$. The proof is analogous to the above.

In order to complete the proof of this lemma we consider $\phi((r_{12} + y_{12})(r_{23} + y_{23}))$. By additivity and the claims, we have

$$\begin{aligned} \phi((r_{12} + y_{12})(r_{23} + y_{23})) &= \phi(r_{12}r_{23}) + \phi(y_{12}y_{23}) \\ (9) \quad &= \phi(r_{12})\phi(r_{23}) - \phi(y_{23})\phi(y_{12}) \\ &\quad \text{(by 3.1 and the denial of the lemma).} \end{aligned}$$

By 2.11 we have either

$$(10) \quad \phi((r_{12} + y_{12})(r_{23} + y_{23})) = \phi(r_{12} + y_{12})\phi(r_{23} + y_{23})$$

or

$$(11) \quad \phi((r_{12} + y_{12})(r_{23} + y_{23})) = -\phi(r_{23} + y_{23})\phi(r_{12} + y_{12}).$$

Combining (10) with (9), we obtain

$$(12) \quad \phi(y_{12})\phi(y_{23}) = -\phi(y_{23})\phi(y_{12}).$$

This gives $y_{12}y_{23} = 0$, a contradiction. Similarly (11) and (9) give

$$(13) \quad \phi(r_{12})\phi(r_{23}) = -\phi(r_{23})\phi(r_{12}).$$

Hence $r_{12}r_{23} = 0$, another contradiction. This completes the proof of the lemma.

3.3. LEMMA. ϕ is a homomorphism from $R_{12} \oplus R_{23} \oplus R_{13}$ into S .

Proof. Since ϕ is additive, it suffices to check the various products $y_{ij}y_{lm}$, $i \neq j$, $l \neq m$.

$$(1) \phi(y_{12}y_{23}) = \phi(y_{12})\phi(y_{23}) \quad (\text{by 3.2}).$$

$$(2) \phi(y_{23}y_{12}) = 0. \text{ But}$$

$$\begin{aligned} \phi(y_{23})\phi(y_{12}) &= [\phi(y_{23}), \phi(y_{12})] + \phi(y_{12})\phi(y_{23}) \\ &= \phi([y_{23}, y_{12}]) + \phi(y_{12})\phi(y_{23}) \\ &= \phi(-y_{12}y_{23}) + \phi(y_{12}y_{23}) \quad (\text{by 3.2}) \\ &= 0. \end{aligned}$$

(3) Each of the other possibilities are trivial since $y_{ij}y_{lm} = 0$ and $\phi(y_{ij})\phi(y_{lm}) = 0$ by 2.8.

3.4. LEMMA. *Let (i, j, k) be a permutation of $(1, 2, 3)$. Then ϕ is a homomorphism of $R_{ij} \oplus R_{jk} \oplus R_{ik}$ into S .*

Proof. We will prove the lemma by showing, if the lemma is true for (i, j, k) , then it is true for (i, k, j) and for (j, i, k) . Noting that these two transpositions generate S_3 , we see then that the lemma is true either for all permutations or for none. By 3.3, we will then be done.

It clearly suffices to let $i = 1, j = 2, k = 3$. Thus ϕ is a homomorphism on $R_{12} \oplus R_{23} \oplus R_{13}$. Suppose there are $x_{13} \in R_{13}$, $x_{32} \in R_{32}$ such that $\phi(x_{13}x_{32}) \neq \phi(x_{13})\phi(x_{32})$. By 2.11, $\phi(x_{13}x_{32}) = -\phi(x_{32})\phi(x_{13})$. Let $x_{23} \in R_{23}$. Then

$$\phi((x_{13}x_{32})x_{23}) = \phi(x_{13}x_{32})\phi(x_{23}) = -\phi(x_{32})\phi(x_{13})\phi(x_{23}) = 0 \quad \text{by 2.8.}$$

Since ϕ is an injection, $x_{13}x_{32}x_{23} = 0$. Hence $x_{13}x_{32}R_{23} = 0$, so $x_{13}x_{32} = 0$. Thus $\phi(x_{13}x_{32}) = 0$. On the other hand, $-\phi(x_{32})\phi(x_{13}) = \phi(x_{13}x_{32}) = \phi([x_{13}, x_{32}]) = [\phi(x_{13}), \phi(x_{32})]$. So $\phi(x_{13})\phi(x_{32}) = 0$. But then $\phi(x_{13}x_{32}) = \phi(x_{13})\phi(x_{32})$, a contradiction.

Now

$$\begin{aligned} \phi(x_{32})\phi(x_{13}) &= [\phi(x_{32}), \phi(x_{13})] + \phi(x_{13})\phi(x_{32}) \\ &= \phi[x_{32}, x_{13}] + \phi(x_{13})\phi(x_{32}) \\ &= -\phi(x_{13}x_{32}) + \phi(x_{13}x_{32}) = 0 = \phi(0) = \phi(x_{32}x_{13}). \end{aligned}$$

ϕ is multiplicative on $x_{32}x_{12}$ and $x_{12}x_{32}$ by 2.8. The argument for the triple $(2, 1, 3)$ is similar to the above.

3.5. LEMMA. *Let $x_{ij}, y_{ij} \in R_{ij}$ and $x_{ji} \in R_{ji}$, $i \neq j$. Then $\phi(x_{ij}x_{ji}y_{ij}) = \phi(x_{ij})\phi(x_{ji})\phi(y_{ij})$.*

Proof. In order to simplify notation, let $i = 1$, and $j = 2$. We will show that

$$[\phi(x_{12}x_{21}y_{12}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})][S, S] = 0.$$

The result will then follow from 2.4. By 2.10 it suffices to show

$$[\phi(x_{12}x_{21}y_{12}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})]\phi(y_{ij}) = 0 \quad \text{for } i \neq j.$$

$$(1) \quad \phi(x_{12}x_{21}y_{12})\phi(z_{12}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(z_{12}) = 0 \quad (\text{by 2.8})$$

$$(2) \quad \phi(x_{12}x_{21}y_{12})\phi(y_{13}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{13}) = 0 \quad (\text{by 2.8})$$

$$\begin{aligned} & \phi(x_{12}x_{21}y_{12})\phi(y_{23}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{23}) \\ (3) \quad &= \phi(x_{12}x_{21}y_{12}y_{23}) - \phi(x_{12})\phi(x_{21})\phi(y_{12}y_{23}) \quad (\text{by 3.4}) \\ &= \phi(x_{12}x_{21}y_{12}y_{23}) - \phi(x_{12})\phi(x_{21}y_{12}y_{23}) \quad (\text{by 3.4}) \\ &= \phi(x_{12}x_{21}y_{12}y_{23}) - \phi(x_{12}x_{21}y_{12}y_{23}) = 0 \quad (\text{by 3.4}). \end{aligned}$$

(4) Letting $y_{21} = \sum_{i=1}^n y_{23}^{(i)} y_{31}^{(i)}$, we then have $\phi(y_{21}) = \sum_{i=1}^n \phi(y_{23}^{(i)})\phi(y_{31}^{(i)})$ by 3.4.

Thus

$$\begin{aligned} & \phi(x_{12}x_{21}y_{12})\phi(y_{21}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{21}) = 0 \quad (\text{by part (3)}). \\ (5) \quad & \phi(x_{12}x_{21}y_{12})\phi(y_{31}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{31}) = 0 - 0 = 0 \quad (\text{by 3.4}). \\ (6) \quad & \phi(x_{12}x_{21}y_{12})\phi(y_{32}) - \phi(x_{12})\phi(x_{21})\phi(y_{12})\phi(y_{32}) = 0 \quad (\text{by 3.4}). \end{aligned}$$

The key to extending ϕ to all of R is given by:

3.6. LEMMA. Let (i, j, k) be any permutation of $(1, 2, 3)$. Suppose that

$$\sum_{s=1}^n x_{ij}^{(s)} x_{ji}^{(s)} = \sum_{t=1}^m x_{ik}^{(t)} x_{ki}^{(t)},$$

then

$$\sum_{s=1}^n \phi(x_{ij}^{(s)})\phi(x_{ji}^{(s)}) = \sum_{t=1}^m \phi(x_{ik}^{(t)})\phi(x_{ki}^{(t)}).$$

Proof. To simplify notation, take $(i, j, k) = (1, 2, 3)$. As before, the method of proof will be to show:

$$\left[\sum_{s=1}^n \phi(x_{12}^{(s)})\phi(x_{21}^{(s)}) - \sum_{t=1}^m \phi(x_{13}^{(t)})\phi(x_{31}^{(t)}) \right] [S, S] = 0.$$

By 2.10 it suffices to verify this for elements of $[S, S]$ of the form $\phi(y_{ij})$, $i \neq j$.

$$\begin{aligned} & \sum_{s=1}^n \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{12}) - \sum_{t=1}^m \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{12}) \\ &= \sum_{s=1}^n \phi(x_{12}^{(s)}x_{21}^{(s)}y_{12}) - \sum_{t=1}^m \phi(x_{13}^{(t)})\phi(x_{31}^{(t)}y_{12}) \quad (\text{by 3.5 and 3.4}) \\ (1) \quad &= \sum_{s=1}^n \phi(x_{12}^{(s)}x_{21}^{(s)}y_{12}) - \sum_{t=1}^m \phi(x_{13}^{(t)}x_{31}^{(t)}y_{12}) \quad (\text{by 3.4}) \\ &= \phi \left(\sum_{s=1}^n x_{12}^{(s)}x_{21}^{(s)}y_{12} - \sum_{t=1}^m x_{13}^{(t)}x_{31}^{(t)}y_{12} \right) \quad (\text{since } \phi \text{ is additive}) \\ &= \phi(0) = 0. \end{aligned}$$

$$\begin{aligned}
 & \sum_{s=1}^n \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{13}) - \sum_{t=1}^m \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{13}) \\
 (2) \quad &= \sum_{s=1}^n \phi(x_{12}^{(s)})\phi(x_{21}^{(s)}x_{13}) - \sum_{t=1}^m \phi(x_{13}^{(t)}x_{31}^{(t)}y_{13}) \quad (\text{by 3.5 and 3.4}) \\
 &= \sum_{s=1}^n \phi(x_{12}^{(s)}x_{21}^{(s)}x_{13}) - \sum_{t=1}^m \phi(x_{13}^{(t)}x_{31}^{(t)}y_{13}) \quad (\text{by 3.4}) \\
 &= \phi(0) = 0 \quad (\text{since } \phi \text{ is additive}).
 \end{aligned}$$

$$(3) \quad \sum_{s=1}^n \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{21}) - \sum_{t=1}^m \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{21}) = 0 - 0 = 0 \quad (\text{by 2.8}).$$

$$(4) \quad \sum_{s=1}^n \phi(x_{12}^{(s)})\phi(x_{21}^{(s)})\phi(y_{23}) - \sum_{t=1}^m \phi(x_{13}^{(t)})\phi(x_{31}^{(t)})\phi(y_{23}) = 0 - 0 = 0 \quad (\text{by 3.4}).$$

(5) The computations for y_{31} and y_{32} are the same as (4).

3.7. COROLLARY. Suppose $i \neq j$ and $\sum_{s=1}^n x_{ij}^{(s)}x_{ji}^{(s)} = 0$. Then $\sum_{s=1}^n \phi(x_{ij}^{(s)})\phi(x_{ji}^{(s)}) = 0$.

Proof. Choose $x_{ik}^{(1)} = x_{ki}^{(1)} = 0$, $k \neq i$, $k \neq j$, and $m = 1$ in 3.6. The result is then immediate.

Corollary 3.7 gives the necessary information to allow us to extend ϕ to all of R . This is done as follows:

3.8. DEFINITION. Let ψ be the mapping of R into S defined by:

- (1) For $x \in R_{ij}$, $i \neq j$, $\psi(x) = \phi(x)$.
- (2) For $x \in R_{ii}$, let $x = \sum_{t=1}^n x_{ij}^{(t)}x_{ji}^{(t)} = \sum_{s=1}^m x_{ik}^{(s)}x_{ki}^{(s)}$ where (i, j, k) is a permutation of $(1, 2, 3)$. Then

$$\psi(x) = \sum_{t=1}^n \phi(x_{ij}^{(t)})\phi(x_{ji}^{(t)}) = \sum_{s=1}^m \phi(x_{ik}^{(s)})\phi(x_{ki}^{(s)}) \quad \text{by 3.6.}$$

- (3) For $x \in R$, let $x = \sum_{i,j=1}^3 x_{ij}$ where $x_{ij} \in R_{ij}$. Then $\psi(x) = \sum_{i,j=1}^3 \psi(x_{ij})$.

3.9. REMARKS. Part (2) of the definition gives a well defined mapping by 3.7. Part (3) is legitimate since the Peirce decomposition is a direct sum. The mapping ψ is an additive mapping of R into S since ϕ is additive on R_{ij} , $i \neq j$, and ψ is by its nature additive on R_{ii} . We hope to show that ψ is the desired extension of ϕ and that it is an associative isomorphism. We begin with,

3.10. LEMMA. ψ is an extension of ϕ to R .

Proof. We must show that $\psi|_{[R,R]} = \phi$. By 2.9 $[R, R]$ is additively generated by elements of the form x_{ij} , $i \neq j$, and by $x_{ik}x_{ki} - x_{ki}x_{ik}$ where $i \neq k$. Thus it suffices to check ψ on elements of this type.

- (1) By definition, $\psi(x_{ij}) = \phi(x_{ij})$ for $i \neq j$.

$$\begin{aligned}
 (2) \quad & \psi(x_{ij}x_{ji} - x_{ji}x_{ij}) = \psi(x_{ij}x_{ji}) - \psi(x_{ji}x_{ij}) \\
 &= \phi(x_{ij})\phi(x_{ji}) - \phi(x_{ji})\phi(x_{ij}) = \phi(x_{ij}x_{ji} - x_{ji}x_{ij}),
 \end{aligned}$$

by the definition of ψ .

3.11. LEMMA. ψ is a homomorphism of R into S .

Proof. Since ψ is already known to be additive, it suffices to show that $\psi(x_{ij}x_{kl}) = \psi(x_{ij})\psi(x_{kl})$ $i, j, k, l = 1, 2, 3$.

(1) $i \neq j, k \neq l, j \neq k$. Then $\psi(x_{ij}x_{kl}) = \psi(0) = 0$. But then $\psi(x_{ij})\psi(x_{kl}) = \phi(x_{ij})\phi(x_{kl})$. But this product is zero by 3.4 if $i = l$, and by 2.8 if $i \neq l$.

(2) $i \neq j, k \neq l, j = k$. If $i = l$, then $\psi(x_{ij}x_{jl}) = \phi(x_{ij})\phi(x_{jl})$ by the definition of ψ . But $\phi(x_{ij})\phi(x_{jl}) = \psi(x_{ij})\psi(x_{jl})$. If $i \neq l$, then $\psi(x_{ij})\psi(x_{jl}) = \phi(x_{ij})\phi(x_{jl}) = \phi(x_{ij}x_{jl}) = \psi(x_{ij}x_{jl})$ by 3.4.

(3) $i = j, k \neq l, i \neq k$. By 3.9 we may assume $x_{ii} = x_{ik}x_{kl}$. Then

$$\psi(x_{ik}x_{kl})\psi(x_{kl}) = \phi(x_{ik})\phi(x_{kl})\phi(x_{kl}) = \phi(x_{ik})0 = 0 = \psi(0) = \psi((x_{ik}x_{kl})x_{kl}).$$

(4) $i = j, k \neq l, i = k$. We may assume $x_{ii} = x_{il}x_{li}$. Then

$$\begin{aligned}\psi(x_{il}x_{li})\psi(y_{il}) &= \phi(x_{il})\phi(x_{li})\phi(y_{il}) = \phi(x_{il})\phi(x_{li})\phi(y_{il}) \\ &= \phi(x_{il}x_{li}y_{il}) = \psi(x_{il}x_{li}y_{il}) = \psi((x_{il}x_{li})x_{kl})\end{aligned}$$

by 3.5.

(5) $i \neq j, k = l$. This case is handled exactly as cases (3) and (4).

(6) $i = j, k = l, i \neq k$. Then we may assume $x_{ii} = x_{ik}x_{kl}$ and $x_{kk} = y_{kl}y_{lk}$, then

$$\psi(x_{ik}x_{kl})\psi(y_{kl}y_{lk}) = \phi(x_{ik})\phi(x_{kl})\phi(y_{kl})\phi(y_{lk}) = 0 \quad (\text{by 2.8}).$$

But $\psi((x_{ik}x_{kl})(y_{kl}y_{lk})) = \psi(0) = 0$.

(7) $i = j, k = l, i = k$. In this case we may assume $x_{ii} = x_{ip}x_{pi}$ and $x_{kk} = y_{ip}y_{pi}$. Then

$$\begin{aligned}\psi(x_{ip}x_{pi})\psi(y_{ip}y_{pi}) &= \phi(x_{ip})\phi(x_{pi})\phi(y_{ip})\phi(y_{pi}) \\ &= \phi(x_{ip}x_{pi}y_{ip})\phi(y_{pi}) = \psi((x_{ip}x_{pi}y_{ip})y_{pi})\end{aligned}$$

by 3.5 and the definition of ψ .

3.12. LEMMA. ψ is an isomorphism of R onto S .

Proof. All that is needed is to show that ψ is a bijection.

(1) ψ is an injection; ψ is nonzero, since ϕ is nonzero. The kernel of ψ is an ideal of the simple ring R . Hence ψ is one-to-one.

(2) ψ is a surjection: Since ψ is an extension of ϕ , $[S, S] \subseteq \text{image of } \psi$. Since the image of a homomorphism is a subring, this gives $[S, S]^- \subseteq \text{im } \psi$. But $[S, S]^- = S$. Hence ψ is onto.

4. The general case. Suppose that assumption 3.1 is false. Then in particular, we have by 2.11:

4.1. ASSUMPTION. There is $r_{12} \in R_{12}$, $r_{23} \in R_{23}$, such that $r_{12}r_{23} \neq 0$ and

$$\phi(r_{12}r_{23}) = -\phi(r_{23})\phi(r_{12}).$$

By using the opposite ring S^* of S , and the results of §3, we will prove the main result.

Let S^* denote the opposite ring of S . Let η denote the canonical anti-isomorphism of S onto S^* . Let $\nu = -\eta$. ν is the negative of an anti-isomorphism of S onto S^* , and hence ν is a Lie isomorphism of S onto S^* . It is clear that $\nu([S, S]) = [S^*, S^*]$. Let κ denote the restriction of ν to $[S, S]$. κ is a Lie isomorphism of $[S, S]$ onto $[S^*, S^*]$. Thus $\kappa \circ \phi$ is a Lie isomorphism of $[R, R]$ onto $[S^*, S^*]$. Furthermore, $\kappa \circ \phi(r_{12}r_{23}) = \kappa(-\phi(r_{23})\phi(r_{12})) = \kappa \circ \phi(r_{12})\kappa \circ \phi(r_{23})$. Thus $\kappa \circ \phi$ is a Lie isomorphism of $[R, R]$ onto $[S, S]$ satisfying 3.1. By 3.13, $\kappa \circ \phi$ can be extended to an associative isomorphism ξ of R onto S . Let $\psi = \nu^{-1} \circ \xi$. ψ is clearly the negative of an anti-isomorphism of R onto S . We must show that ψ extends ϕ . Let $[r_1, r_2] \in [R, R]$.

$$\psi([r_1, r_2]) = \nu^{-1} \circ \xi([r_1, r_2]) = \nu^{-1} \circ \kappa \circ \phi[r_1, r_2] = \kappa^{-1} \circ \kappa \circ \phi[r_1, r_2] = \phi[r_1, r_2].$$

We have now completed the proof of:

4.2. MAIN THEOREM. *Let R be a simple ring containing three nonzero orthogonal idempotents $\{e_i\}_{i=1}^3$ such that $1 = \sum_{i=1}^3 e_i$. Let S be a simple ring with 1. If ϕ is a Lie isomorphism of $[R, R]$ onto $[S, S]$, and the characteristic of R is not two or three, then ϕ can be extended to an additive bijection ψ of R onto S such that ψ is either an associative isomorphism or the negative of an anti-isomorphism of R onto S .*

We give now two corollaries of the main result. The first is the classical result, which is due to Landherr [4].

4.3. COROLLARY. *Let F_n be the $n \times n$ matrices over F , a field of characteristic 0. Let G_m be the $m \times m$ matrices over G , a field of characteristic 0. Suppose either $n \geq 3$ or $m \geq 3$. If ϕ is a Lie isomorphism of $[F_n, F_n]$ onto $[G_m, G_m]$ then ϕ may be extended to a mapping ψ of R onto S such that ψ is either an associative isomorphism or the negative of an anti-isomorphism of F_n onto G_m .*

Proof. The corollary follows immediately from 4.2, by using $e_i = e_{ii}$, the standard matrix units.

4.4. COROLLARY (MARTINDALE [5]). *Let R, S be simple rings of characteristic not 2 or 3. Suppose R contains three orthogonal idempotents $\{e_i\}_{i=1}^3$ such that $\sum_{i=1}^3 e_i = 1$. Let ϕ be a Lie isomorphism of R onto S . Then $\phi = \sigma + \tau$, where σ is either an isomorphism of R onto S or the negative of an anti-isomorphism of R onto S , and τ is an additive mapping of R into $Z(S)$, the center of S , such that τ maps $[R, R]$ to zero.*

Proof. Let η be the restriction of ϕ to $[R, R]$. It is clear that $\eta[R, R] = [S, S]$, and so η is a Lie isomorphism of $[R, R]$ onto $[S, S]$. Let σ be the extension of η to R guaranteed by 4.2. Let $\tau = \phi - \sigma$. τ is an additive mapping of R into S , since both ϕ and σ are additive. Furthermore, $\tau[x, y] = (\phi - \sigma)([x, y]) = \phi[x, y] - \sigma[x, y] = \eta[x, y] - \eta[x, y] = 0$. It remains to show that τ maps R into the center of S . That is we wish to show that $[\tau(x), S] = 0$ for all $x \in R$. Since $[S, S]^- = S$, it suffices to show that $[\tau(x), [S, S]] = 0$ for all $x \in R$. But η is a surjection of $[R, R]$

onto $[S, S]$, hence it suffices to show $[\tau(x), \eta(y)] = 0$ for all $x \in R$ and for all $y \in [R, R]$. But

$$\begin{aligned} [\tau(x), \eta(y)] &= [\phi(x) - \sigma(x), \eta(y)] = [\phi(x), \eta(y)] - [\sigma(x), \eta(y)] \\ &= [\phi(x), \phi(y)] - [\sigma(x), \sigma(y)] = \phi[x, y] - \sigma[x, y] \\ &= (\phi - \sigma)[x, y] = 0, \end{aligned}$$

since both ϕ and σ are Lie isomorphisms.

By definition, $\phi = \sigma + \tau$, and the proof is complete.

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